



# Law of large numbers for non-additive measures

Yann Rébillé

Institut d'Economie et de Management de Nantes-IAE, LEMNA, chemin la Censive du Tertre, BP 62232, 44322 Nantes Cedex 3, France

## ARTICLE INFO

### Article history:

Received 13 September 2007

Available online 27 November 2008

Submitted by M. Peligrad

### Keywords:

Law of large numbers

Exact games

Multi-prior model

## ABSTRACT

Our aim is to establish law of large numbers for classes of non-additive measures. For balanced games we obtain weak and strong law of large numbers for bounded random variables. A sharper result is obtained for exact games. We also provide an extension to upper envelope measures.

© 2008 Elsevier Inc. All rights reserved.

## 1. Introduction

Non-additive measures are nowadays studied in different fields of expertise. The first systematic treatment of non-additive measures starts traces back to Choquet's seminal contribution in potential theory [4], where non-additive measures are termed capacities. From the historical perspective, non-additive set-functions were found useful in three fields of study. In fuzzy set theory, as fuzzy measures, they are studied in connection to non-statistical uncertainty [16,20]. In Bayesian statistics, as imprecise probabilities, they allow for a better handling of uncertainty. In mathematical economics, as games with transferable utility, they are introduced in cooperative game theory and more recently in mathematical finance as risk measures.

Since Kolmogorov's [10] axiomatic treatise on probability, the measure theoretic approach became the standard framework.  $\sigma$ -Additive measures turned out to be the appropriate objects to model random phenomena. A major requirement for any probability theory is to be able to give a frequentist justification to probability numbers via limit frequencies. Stated in an equivalent manner, laws of large numbers (LLN) should hold for any theory aiming at modelling uncertainty. Our aim is to establish LLN for classes of non-additive measures.

Various generalizations of the SLLN can be found in the works of Puri and Ralescu [14] (see also the references therein). In [14] (see also [5]), SLLN is established for random sets (taking values in Banach spaces) instead of random variables. In [15], SLLN is established with respect to a set-valued measure instead of a single-valued measure. A typical example is the model of "interval of measures" of de Robertis and Hartigan [6], that deals with a set of measures bounded by two measures. However, despite its resemblance, their approaches are different from ours. We shall deal with probability measures bounded by a given non-additive measure, i.e. a non-necessarily additive set-function. Our approach is closer to Markov's conditions for SLLN and departures from the earlier references and other existing topological approaches for non-additive measures such as Marinacci [12] for compact spaces and Maccheroni and Marinacci [11] for Polish spaces where powerful analytical methods are used.

An important class of non-additive measures which contains some very mild additivity conditions is the one of balanced games [2,17] and with more structure, exact games [18]. These games are particularly important since they introduce a key concept to understand the geometry of a game: the core, i.e., the set of measures dominated by the game. A natural

E-mail address: Yann.Rebille@univ-nantes.fr.

approach is to consider non-additive versions of Markov's conditions in order to establish weak and strong law of large numbers for balanced and exact games. Our results can be extended through upper integrals. The interest in upper integrals relies on the possibility to deal directly with a set of measures such as the core. This gives a more flexible treatment of uncertainty as in the multi-prior model of Gilboa and Schmeidler ([9], see [3] for  $\sigma$ -measures).

## 2. Definitions

Let  $(\Omega, \mathcal{A})$  be a measurable space. A set-function  $w : \mathcal{A} \rightarrow \mathbb{R}^+$  is called a *game* if  $w(\emptyset) = 0$ .

$w$  is said to be *monotone* if  $w(A) \geq w(B)$  whenever  $A \supset B$ .

$w$  is said to be *subadditive* if  $w(A \cup B) \leq w(A) + w(B)$  for all  $A, B \in \mathcal{A}$  with  $A \cap B = \emptyset$ .

If the inequality is replaced by an equality  $w$  is *additive*, i.e.  $w \in \mathbf{ba}^+$ . If moreover,  $w(\bigcup_n A_n) = \sum_n w(A_n)$  holds for any countable sequence of disjoint sets,  $w$  is called  *$\sigma$ -additive*, i.e.,  $w \in \mathbf{ca}^+$ .

A monotone subadditive game is called a *submeasure*.

The *conjugate* of  $w$  denoted by  $w_c$  is defined by  $w_c(A) = w(\Omega) - w(A^c)$  for all  $A \in \mathcal{A}$ .

$w$  is said to be *continuous from above* (below) if for all  $A_n \downarrow (\uparrow) A$ ,  $w(A_n) \downarrow (\uparrow)$ ,  $w(A)$ .  $w$  is *order-continuous* if  $w(A_n) \downarrow 0$  whenever  $A_n \downarrow \emptyset$ .<sup>1</sup>

For submeasures order-continuity is an equivalent condition for continuity from above and below.

**Proposition 2.1.** *Let  $w$  be a submeasure. Then,  $w$  is continuous from above and from below if and only if  $w$  is order-continuous.*

**Proof.** (only if) follows by definition.

(if) Let  $A_n \uparrow A$ . By monotonicity and subadditivity we get,  $w(A_n) \leq w(A) \leq w(A_n) + w(A \setminus A_n)$ , since  $A \setminus A_n \downarrow \emptyset$ , order continuity entails  $\lim_n w(A_n) \leq w(A) \leq \lim_n w(A_n)$ .

Similarly for  $A_n \downarrow A$ . We get,  $w(A) \leq w(A_n) \leq w(A) + w(A_n \setminus A)$  thus  $w(A) \leq \lim_n w(A_n) \leq w(A)$ .  $\square$

Denote with  $AC(w)$  the *anti-core*<sup>2</sup> of  $w$  given by

$$AC(w) = \{P : P \in \mathbf{ba}^+, P \leq w, P(\Omega) = w(\Omega)\}.$$

If  $AC(w) \neq \emptyset$ ,  $w$  is called *balanced* (Bondareva [2,17]).

Moreover if for all  $A \in \mathcal{A}$ ,  $w(A) = \max\{P(A) : P \in AC(w)\}$ , then  $w$  is called *exact*<sup>3</sup> [18].

When  $w$  is order continuous (thus continuous as a submeasure) then any element of  $AC(w)$  is  $\sigma$ -additive. A partial converse holds:

**Theorem.** (See Schmeidler [18, Theorem 3.2, p. 219].) *Let  $w$  be an exact game, then  $AC(w) \subset \mathbf{ca}^+$  if and only if  $w$  is order continuous.*

The Choquet integral will play the rôle of the standard Lebesgue integral for usual measures. A real function  $X : \Omega \rightarrow \mathbb{R}$  is *measurable* if  $\{X > t\} = \{\omega : X(\omega) > t\} \in \mathcal{A}$  for all  $t \in \mathbb{R}$ . We denote by  $\mathbb{B}(\Omega, \mathcal{A})$  the space of bounded  $\mathcal{A}$ -measurable functions ( $\mathbb{B}$  for short). For  $X \in \mathbb{B}$ ,  $X \geq 0$ , its *Choquet integral* [4,19] with respect to  $w$  is given by

$$\int X dw = \int_0^{+\infty} w(\{X > t\}) dt,$$

where the strict inequality can be replaced by a large inequality. For real-valued functions we extended it as the *asymmetric integral*,

$$\int X dw = \int_0^{+\infty} w(\{X > t\}) dt + \int_{-\infty}^0 w(\{X > t\}) - w(\Omega) dt.$$

The properties of the Choquet integral functional are exposed in [8,13].

Whenever  $w$  is balanced, one can introduce the *upper* (*lower*) *integral* of  $X \in \mathbb{B}$ , given by

$$J_w(X) = \sup_{P \in AC(w)} \int X dP, \quad I_w(X) = \inf_{P \in AC(w)} \int X dP;$$

by construction,  $\int(\cdot) dw_c \leq I_w \leq J_w \leq \int(\cdot) dw$ , since  $J_w(X) = -I_w(-X)$  and  $\int X dw = -\int -X dw_c$ .

<sup>1</sup> If  $w$  is additive then  $\sigma$ -additivity, continuity from below, continuity from above and order-continuity are equivalent.

<sup>2</sup> The core is defined as  $C(w_c) = \{P : P \in \mathbf{ba}^+, P \geq w_c, P(\Omega) = w_c(\Omega)\}$ , and coincide with  $AC(w)$ .

<sup>3</sup> Generally speaking  $w$  should be called *anti-exact* since it is the conjugate  $w_c$  which is exact, i.e.,  $w_c(A) = \min\{P(A) : P \in C(w_c)\}$ . However properly speaking it is the core which is exact and not the game.

### 3. Law of large numbers

In order to consider Markov's conditions we need to define an analogous for the variance and the covariance for balanced games. Let  $X, Y \in \mathbb{B}$ , the covariance is given by

$$\text{cov}_w(X, Y) = \sup_{AC(w)} \text{cov}_P(X, Y),$$

and the variance of  $X$  is given by

$$V_w(X) = \sup_{AC(w)} V_P(X).$$

$X, Y$  are said to be  $w$ -negatively correlated if  $\text{cov}_w(X, Y) \leq 0$ .

#### 3.1. Weak law of large numbers

We now establish a non-additive version for balanced games of the classical weak law of large numbers, i.e. Bienaymé–Tchebitchev's theorem. The result can be sharpen for exact games.

**Theorem 3.1.** Let  $w$  be a balanced game and a sequence  $\{X_n\}_n \subset \mathbb{B}$ . Assume  $\{V_w(\frac{1}{n} \sum_{k=1}^n X_k)\}_n$  converges to 0. If  $AC(w) \cap \mathbf{ca}^+ \neq \emptyset$  then for all  $\epsilon > 0$ ,

$$\lim_n w \left( \left\{ \frac{1}{n} \sum_{k=1}^n \int X_k dw_c - \epsilon \leq \frac{1}{n} \sum_{k=1}^n X_k \leq \frac{1}{n} \sum_{k=1}^n \int X_k dw + \epsilon \right\} \right) = w(\Omega).$$

Moreover, if  $w$  is order-continuous and exact then

$$\lim_n w \left( \left\{ \frac{1}{n} \sum_{k=1}^n X_k < \frac{1}{n} \sum_{k=1}^n \int X_k dw_c - \epsilon \right\} \cup \left\{ \frac{1}{n} \sum_{k=1}^n X_k > \frac{1}{n} \sum_{k=1}^n \int X_k dw + \epsilon \right\} \right) = 0.$$

In particular if  $w = P$  is  $\sigma$ -additive and the  $X_n$ 's have common mean then

$$\lim_n P \left( \left\{ \frac{1}{n} \sum_{k=1}^n X_k \in \left[ \int X_1 dP - \epsilon, \int X_1 dP + \epsilon \right] \right\} \right) = P(\Omega).$$

**Proof.** We shall prove in fact a sharper result using the upper and lower integrals instead of the Choquet integrals.

The proof relies directly on the application of Bienaymé–Tchebitchev's inequality for classical probabilities. Let  $Y_n = X_n - \int X_n dP$  for all  $n$ . For  $P \in AC(w) \cap \mathbf{ca}^+$ ,

$$\frac{1}{n^2} V_P \left( \sum_{k=1}^n Y_k \right) = \frac{1}{n^2} V_P \left( \sum_{k=1}^n X_k \right) \leq \frac{1}{n^2} V_w \left( \sum_{k=1}^n X_k \right) \rightarrow 0 \quad (n \rightarrow +\infty)$$

thus for  $\epsilon > 0$  we get

$$\lim_n P \left( \left\{ \frac{1}{n} \sum_{k=1}^n \int X_k dP - \epsilon \leq \frac{1}{n} \sum_{k=1}^n X_k \leq \frac{1}{n} \sum_{k=1}^n \int X_k dP + \epsilon \right\} \right) = P(\Omega).$$

Now since  $I_w(X_n) \leq \int X_n dP \leq J_w(X_n)$  for all  $n$  we can conclude

$$\lim_n w \left( \left\{ \frac{1}{n} \sum_{k=1}^n I_w(X_k) - \epsilon \leq \frac{1}{n} \sum_{k=1}^n X_k \leq \frac{1}{n} \sum_{k=1}^n J_w(X_k) + \epsilon \right\} \right) = w(\Omega).$$

For the second part of the theorem we use a powerful result for continuous exact games (see [7]). According to Theorem 10 in [13, p. 11] and its remark on positive games, there exists a measure  $\lambda \in AC(w)$  for which the measure in  $AC(w)$  are uniformly countably additive, i.e.

$$\forall \eta > 0, \exists \delta(\eta) > 0 \text{ such that } \lambda(A) < \delta(\eta) \Rightarrow P(A) < \eta \text{ for all } A \in \mathcal{A}, P \in AC(w).$$

Put

$$A_n = \left\{ \frac{1}{n} \sum_{k=1}^n X_k < \frac{1}{n} \sum_{k=1}^n I_w(X_k) - \epsilon \right\} \cup \left\{ \frac{1}{n} \sum_{k=1}^n X_k > \frac{1}{n} \sum_{k=1}^n J_w(X_k) + \epsilon \right\}.$$

Let  $\eta = \frac{1}{p}$  for  $p \in \mathbb{N}$  and its corresponding  $\delta(p)$ . According to the first part of the theorem we have  $\lim_n \lambda(A_n) = 0$ , so there exists  $N(p)$  such that for all  $n \geq N(p)$ ,  $\lambda(A_n) < \delta(p)$ . Thus for all  $P \in AC(w)$  we have  $P(A_n) < \frac{1}{p}$ , and  $w(A_n) < \frac{1}{p}$  follows by exactness.  $\square$

**Remark 1.** If for some (resp. all)  $P \in AC(w) \cap \mathbf{ca}^+ (\subset \mathbf{ca}^+)$ ,  $\{V_P(\frac{1}{n} \sum_{k=1}^n X_k)\}_n$  converges to 0 then the first (second) part of conclusion holds.

### 3.2. Strong law of large numbers

We now establish a non-additive version for balanced games of the classical strong law of large numbers, i.e., Markov's theorem. For exact games the result can be made more precise.

**Theorem 3.2.** Let  $w$  be a balanced game and a sequence  $\{X_n\}_n \subset \mathbb{B}$ . Assume  $\{V_w(X_n)\}_n$  and  $\{nV_w(\frac{1}{n} \sum_{k=1}^n X_k)\}_n$  are bounded. If  $AC(w) \cap \mathbf{ca}^+ \neq \emptyset$ ,

$$w\left(\left\{\liminf_n \frac{1}{n} \sum_{k=1}^n \int X_n dw_c \leq \liminf_n \frac{1}{n} \sum_{k=1}^n X_k \leq \limsup_n \frac{1}{n} \sum_{k=1}^n X_k \leq \limsup_n \frac{1}{n} \sum_{k=1}^n \int X_n dw\right\}\right) = w(\Omega).$$

Moreover, if  $w$  is order-continuous and exact then

$$w\left(\left\{\liminf_n \frac{1}{n} \sum_{k=1}^n X_k < \liminf_n \frac{1}{n} \sum_{k=1}^n \int X_n dw_c\right\} \cup \left\{\limsup_n \frac{1}{n} \sum_{k=1}^n X_k > \limsup_n \frac{1}{n} \sum_{k=1}^n \int X_n dw\right\}\right) = 0.$$

In particular if  $w = P$  is  $\sigma$ -additive and the  $X_n$ 's have common mean then

$$w\left(\left\{\lim_n \frac{1}{n} \sum_{k=1}^n X_k = \int X_1 dP\right\}\right) = P(\Omega).$$

**Proof.** We shall prove in fact a sharper result using the upper and lower integrals instead of the Choquet integrals.

We may assume that  $\limsup_n \frac{1}{n} \sum_{k=1}^n J_w(X_n) < +\infty$  or  $-\infty < \liminf_n \frac{1}{n} \sum_{k=1}^n I_w(X_n)$ , otherwise the statement is immediate. Let us assume that  $\limsup_n \frac{1}{n} \sum_{k=1}^n J_w(X_n) < +\infty$ , otherwise we work with  $-X_n$ .

Put  $Y_n = X_n - \int X_n dP$  for all  $n$ . For  $P \in AC(w) \cap \mathbf{ca}^+$ , we have

$$V_P(Y_n) = V_P(X_n) \leq V_w(X_n) \leq D,$$

and also,

$$nV_P\left(\frac{1}{n} \sum_{k=1}^n Y_k\right) = nV_P\left(\frac{1}{n} \sum_{k=1}^n X_k\right) \leq nV_w\left(\frac{1}{n} \sum_{k=1}^n X_k\right) \leq D'.$$

We can apply the law of large number for the classical case

$$P\left(\left\{\limsup_n \frac{1}{n} \sum_{k=1}^n Y_k \leq 0\right\}\right) = P(\Omega) = w(\Omega).$$

Now since  $\limsup_n$  is a subadditive functional we have for any sequences  $\{a_n\}_n, \{b_n\}_n \subset \mathbb{R}$ ,  $\limsup_n a_n + b_n \geq \limsup_n a_n + \liminf_n b_n \geq \liminf_n a_n + b_n$ , thus

$$P\left(\left\{\limsup_n \frac{1}{n} \sum_{k=1}^n X_k + \liminf_n -\frac{1}{n} \sum_{k=1}^n \int X_k dP \leq 0\right\}\right) = P(\Omega) = w(\Omega)$$

that is

$$P\left(\left\{\limsup_n \frac{1}{n} \sum_{k=1}^n X_k \leq \limsup_n \frac{1}{n} \sum_{k=1}^n \int X_k dP\right\}\right) = P(\Omega) = w(\Omega)$$

and finally

$$P\left(\left\{\limsup_n \frac{1}{n} \sum_{k=1}^n X_k \leq \limsup_n \frac{1}{n} \sum_{k=1}^n J_w(X_k)\right\}\right) = P(\Omega) = w(\Omega).$$

Similarly,

$$P\left(\left\{\liminf_n \frac{1}{n} \sum_{k=1}^n Y_k \geq 0\right\}\right) = P(\Omega) = w(\Omega).$$

Following the same line we get

$$P\left(\left\{\liminf_n \frac{1}{n} \sum_{k=1}^n X_k \geq \liminf_n \frac{1}{n} \sum_{k=1}^n I_w(X_k)\right\}\right) = P(\Omega) = w(\Omega).$$

Combining these results it follows

$$P\left(\left\{\limsup_n \frac{1}{n} \sum_{k=1}^n X_k \leq \limsup_n \frac{1}{n} \sum_{k=1}^n J_w(X_k)\right\} \cap \left\{\liminf_n \frac{1}{n} \sum_{k=1}^n X_k \geq \liminf_n \frac{1}{n} \sum_{k=1}^n I_w(X_k)\right\}\right) = P(\Omega) = w(\Omega);$$

the conclusion follows since  $P \leq w$ .

For the second part of the theorem. For any  $P \in AC(w) \subset ca$  it holds

$$P\left(\left\{\limsup_n \frac{1}{n} \sum_{k=1}^n X_k > \limsup_n \frac{1}{n} \sum_{k=1}^n J_w(X_k)\right\} \cup \left\{\liminf_n \frac{1}{n} \sum_{k=1}^n X_k < \liminf_n \frac{1}{n} \sum_{k=1}^n I_w(X_k)\right\}\right) = 0;$$

by exactness it comes

$$w\left(\left\{\limsup_n \frac{1}{n} \sum_{k=1}^n X_k > \limsup_n \frac{1}{n} \sum_{k=1}^n J_w(X_k)\right\} \cup \left\{\liminf_n \frac{1}{n} \sum_{k=1}^n X_k < \liminf_n \frac{1}{n} \sum_{k=1}^n I_w(X_k)\right\}\right) = 0. \quad \square$$

**Remark 2.** If for some (resp. all)  $P \in AC(w) \cap \mathbf{ca}^+ (\subset \mathbf{ca}^+)$ ,  $\{V_P(X_n)\}_n$  and  $\{nV_P(\frac{1}{n} \sum_{k=1}^n X_k)\}_n$  are bounded then the first (second) part of conclusion holds.

Another sufficient condition to obtain the required conditions is when the  $\{X_n\}_n$  are pairwise  $w$ -negatively dependent random variables or  $w$ -negatively correlated.

Two measurable functions  $X, Y$  are *pairwise  $w$ -negatively dependent* if for all  $x, y > 0$  and for all  $P \in AC(w)$

$$P(\{X > x\} \cap \{Y > y\}) \leq P(\{X > x\})P(\{Y > y\}).$$

A sequence  $\{X_n\}_n$  of measurable functions is *pairwise  $w$ -negatively dependent (correlated)* if for all  $n, m \in \mathbb{N}$   $X_n$  and  $X_m$  are  $w$ -negatively dependent (correlated). By definition  $w$ -negatively dependent functions are  $w$ -negatively correlated.

**Theorem 3.3.** Let  $\{X_n\}_n$  be a sequence of  $w$ -negatively correlated (or  $w$ -negatively dependent) measurable functions.

The conclusions of Theorem 3.1, respectively 3.2 remain valid whenever  $\{\frac{1}{n} \sum_{k=1}^n V_w(X_n)\}_n$  is bounded, respectively  $\{V_w(X_n)\}_n$  is bounded.

**Proof.** Let  $\{X_n\}_n$  be  $w$ -negatively correlated and  $P \in AC(w)$ . Then  $\{X_n\}_n$  is  $P$ -negatively correlated. We prove by induction that

$$V_P\left(\sum_{k=1}^n X_k\right) \leq \sum_{k=1}^n V_P(X_k).$$

For  $n = 1$  it is immediate. For  $n > 1$ ,

$$\begin{aligned} V_P\left(\sum_{k=1}^n X_k\right) &= V_P\left(\sum_{k=1}^{n-1} X_k\right) + V_P(X_n) + 2 \operatorname{cov}_P\left(X_n, \sum_{k=1}^{n-1} X_k\right) \\ &= V_P\left(\sum_{k=1}^{n-1} X_k\right) + V_P(X_n) + 2 \sum_{k=1}^{n-1} \operatorname{cov}_P(X_n, X_k) \\ &\leq V_P\left(\sum_{k=1}^{n-1} X_k\right) + V_P(X_n) \\ &\leq \sum_{k=1}^n V_P(X_k). \end{aligned}$$

If  $\{\frac{1}{n} \sum_{k=1}^n V_w(X_k)\}_n$  is bounded by  $D$ , then

$$\begin{aligned} V_P\left(\frac{1}{n} \sum_{k=1}^n X_k\right) &= \frac{1}{n^2} V_P\left(\sum_{k=1}^n X_k\right) \leq \frac{1}{n^2} \sum_{k=1}^n V_P(X_k) \\ &\leq \frac{1}{n} \frac{1}{n} \sum_{k=1}^n V_w(X_k) \leq \frac{1}{n} D \rightarrow 0 \quad (n \rightarrow +\infty). \end{aligned}$$

If  $\{V_w(X_n)\}_n$  is bounded by  $D$  then  $V_P(X_n) \leq V_w(X_n) \leq D$  and we have,  $V_P(\sum_{k=1}^n X_k) \leq nD$  thus

$$nV_P\left(\frac{1}{n} \sum_{k=1}^n X_k\right) \leq D. \quad \square$$

#### 4. Extension to upper integrals

Upper integrals are a natural generalization of the classical Lebesgue integral, for they consider a set of probabilities instead of a sole one. This situation is encountered in a framework of decision making under uncertainty as the multi-prior model [9]. Typically, we consider the functional

$$J_{\mathcal{C}} : \mathbb{B} \rightarrow \mathbb{R} : X \mapsto \max \left\{ \int X dP : P \in \mathcal{C} \right\},$$

where  $\mathcal{C}$  is a non-empty convex weak-star compact set of finitely additive probabilities, i.e.  $\mathcal{C} \subset \mathbf{ba}_1^+$ . The associated game is defined by  $w_{\mathcal{C}}(A) = \max\{P(A) : P \in \mathcal{C}\}$ . By construction this game is exact since  $\mathcal{C} \subset AC(w_{\mathcal{C}})$ .

The upper integral introduced for balanced games is a special case where  $J_w = J_{AC(w)}$ . For all  $X \in \mathbb{B}$ ,

$$\int X d(w_{\mathcal{C}})_c \leq I_{\mathcal{C}}(X) \leq J_{\mathcal{C}}(X) \leq \int X dw_{\mathcal{C}},$$

where  $I_{\mathcal{C}}(X) = \min\{\int X dP : P \in \mathcal{C}\}$ .

Theorems 3.1–3.3 can be directly adapted<sup>4</sup> with the upper envelope functional instead of the Choquet functional provided  $\mathcal{C} \cap \mathbf{ca}^+ \neq \emptyset$  or  $\mathcal{C} \subset \mathbf{ca}^+$  and the convergence statement is expressed with respect to  $w_{\mathcal{C}}$  instead of  $w$ . Moreover Remarks 1 and 2 are true if we consider the set  $\mathcal{C}$  instead of  $AC(w)$ .

**Theorem 4.1.** Let  $\mathcal{C}$  be a non-empty subset of  $\mathbf{ba}^+$ ,  $w_{\mathcal{C}}$  its associated submeasure with  $w_{\mathcal{C}}(\Omega) < \infty$  and a sequence  $\{X_n\}_n \subset \mathbb{B}$ . Assume  $\{V_{\mathcal{C}}(\frac{1}{n} \sum_{k=1}^n X_k)\}_n$  converges to 0. If  $\mathcal{C} \cap \mathbf{ca}^+ \neq \emptyset$  then for all  $\epsilon > 0$ ,

$$\lim_n w_{\mathcal{C}} \left( \left\{ \frac{1}{n} \sum_{k=1}^n I_{\mathcal{C}}(X_k) - \epsilon \leq \frac{1}{n} \sum_{k=1}^n X_k \leq \frac{1}{n} \sum_{k=1}^n J_{\mathcal{C}}(X_k) + \epsilon \right\} \right) = w_{\mathcal{C}}(\Omega).$$

Moreover, if  $\mathcal{C} \subset \mathbf{ca}^+$  and  $\mathcal{C}$  is convex and weak-star compact then

$$\lim_n w_{\mathcal{C}} \left( \left\{ \frac{1}{n} \sum_{k=1}^n X_k < \frac{1}{n} \sum_{k=1}^n I_{\mathcal{C}}(X_k) - \epsilon \right\} \cup \left\{ \frac{1}{n} \sum_{k=1}^n X_k > \frac{1}{n} \sum_{k=1}^n J_{\mathcal{C}}(X_k) + \epsilon \right\} \right) = 0.$$

Similarly,

**Theorem 4.2.** Let  $\mathcal{C}$  be a non-empty subset of  $\mathbf{ba}^+$ ,  $w_{\mathcal{C}}$  its associated submeasure with  $w_{\mathcal{C}}(\Omega) < \infty$  and a sequence  $\{X_n\}_n \subset \mathbb{B}$ . Assume  $\{V_{\mathcal{C}}(X_n)\}_n$  and  $\{nV_{\mathcal{C}}(\frac{1}{n} \sum_{k=1}^n X_k)\}_n$  are bounded. If  $\mathcal{C} \cap \mathbf{ca}^+ \neq \emptyset$  then

$$w_{\mathcal{C}} \left( \left\{ \liminf_n \frac{1}{n} \sum_{k=1}^n I_{\mathcal{C}}(X_k) \leq \liminf_n \frac{1}{n} \sum_{k=1}^n X_k \leq \limsup_n \frac{1}{n} \sum_{k=1}^n X_k \leq \limsup_n \frac{1}{n} \sum_{k=1}^n J_{\mathcal{C}}(X_k) \right\} \right) = w_{\mathcal{C}}(\Omega).$$

Moreover, if  $\mathcal{C} \subset \mathbf{ca}^+$  and  $\mathcal{C}$  is weak-star compact then

$$w_{\mathcal{C}} \left( \left\{ \liminf_n \frac{1}{n} \sum_{k=1}^n X_k < \liminf_n \frac{1}{n} \sum_{k=1}^n I_{\mathcal{C}}(X_k) \right\} \cup \left\{ \limsup_n \frac{1}{n} \sum_{k=1}^n X_k > \limsup_n \frac{1}{n} \sum_{k=1}^n J_{\mathcal{C}}(X_k) \right\} \right) = 0.$$

<sup>4</sup> The covariance, variance, negative correlation are defined with respect to  $\mathcal{C}$  instead of  $AC(w)$ .

The following proposition is essentially built upon the proof of Lemma 9 in [13] and makes precise the relation between continuity and set of measures.

**Proposition 4.1.** *Let  $\mathcal{C} \subset \mathbf{ba}^+$  be a non-empty weak-star compact set of non-negative charges and  $w_{\mathcal{C}} = \max_{\mathcal{C}} P$  the associated exact game. Then  $\mathcal{C} \subset \mathbf{ca}^+$  if and only if  $w_{\mathcal{C}}$  is continuous.*

**Proof.** (if) It is immediate.

(only if) Since  $\mathcal{C}$  is weak-star compact and  $\mathcal{C} \subset \mathcal{AC}(w_{\mathcal{C}})$ ,  $w_{\mathcal{C}}$  is exact. From Proposition 2.1 it remains to prove that  $w_{\mathcal{C}}$  is continuous at  $\emptyset$ . Let  $\{A_n\}_n \subset \mathcal{A}$  be a weakly decreasing sequence converging to  $\emptyset$ . Consider the functions

$$\phi_n : \mathcal{C} \rightarrow [0, 1]: \mu \mapsto \mu(A_n).$$

$\{\phi_n\}_n$  is a weakly decreasing sequence of weak-star continuous functions converging to 0. Thus by Dini's theorem, the convergence must be uniform, i.e. for all  $\epsilon > 0$  there exists  $n(\epsilon)$  such that for  $n \geq n(\epsilon)$ ,  $\sup_{\mathcal{C}} \{\mu(A_n)\} < \epsilon$ , thus  $w_{\mathcal{C}}(A_n) < \epsilon$ .  $\square$

**Remark 3.** Theorem 3.2 in [18] can be seen as the special case where  $\mathcal{C} = \mathcal{AC}(w)$ . Since  $\mathcal{AC}(w)$  is weak-star compact, if  $\mathcal{AC}(w) \subset \mathbf{ca}^+$  then  $w_{\mathcal{AC}(w)}$  is continuous, if moreover  $w$  is exact i.e.,  $w = w_{\mathcal{AC}(w)}$ , then  $w$  is continuous. Moreover, since  $w_{\mathcal{C}}$  is continuous whenever  $\mathcal{C} \subset \mathbf{ca}^+$  it follows that  $\mathcal{AC}(w_{\mathcal{C}}) \subset \mathbf{ca}^+$ .

In order to prove the second part of the weak law of large numbers we need some preliminary material.

**Definition 4.1.** Let  $P \in \mathbf{ca}^+$ ,  $\mathcal{C} \subset \mathbf{ca}^+$ .  $P$  weakly dominates  $\mathcal{C}$ , i.e.  $P \gg_{wk} \mathcal{C}$ , if  $\forall A \in \mathcal{A}$ ,  $\forall Q \in \mathcal{C}$ ,  $P(A) = 0 \Rightarrow Q(A) = 0$ ,  $P$  uniformly dominates  $\mathcal{C}$ , i.e.  $P \gg_u \mathcal{C}$ , if  $\forall \epsilon > 0$ ,  $\exists \eta > 0$ , s.t.  $P(A) < \eta \Rightarrow Q(A) < \epsilon$ ,  $\forall Q \in \mathcal{C}$ .

Whenever  $\mathcal{C} = \{Q\}$  then both definitions coincide, the equivalence can be maintained if  $\mathcal{C}$  is weak-star compact,

**Proposition 4.2.** *Let  $P \in \mathbf{ca}^+$  and  $\mathcal{C}$  a weak-star compact subset of  $\mathbf{ca}^+$ . The following statements are equivalent,*

- (i)  $P \gg_{wk} \mathcal{C}$ ,
- (ii)  $P \gg_u \mathcal{C}$ ,
- (iii)  $\forall \{A_n\}_n \subset \mathcal{A}$ ,  $P(A_n) \rightarrow 0 \Rightarrow w_{\mathcal{C}}(A_n) \rightarrow 0$ .

**Proof.** (ii)  $\Rightarrow$  (i) is immediate.

(iii) is a mere reformulation of (ii), since (ii)  $\Rightarrow$  (iii) is immediate.

(iii)  $\Rightarrow$  (ii). Assume (ii) does not hold.

There exists  $\bar{\epsilon} > 0$ , such that for all  $\eta = \frac{1}{n}$ , there exists  $A_n \in \mathcal{A}$  and  $Q_n \in \mathcal{C}$  such that  $P(A_n) \leq \frac{1}{n}$  and  $Q_n(A_n) \geq \bar{\epsilon}$ , thus  $w_{\mathcal{C}}(A_n) \geq \bar{\epsilon}$ .

(i)  $\Rightarrow$  (iii). Assume (iii) does not hold.

There exists  $A_n \in \mathcal{A}$  such that  $P(A_n) \rightarrow 0$  and  $w_{\mathcal{C}}(A_n) \not\rightarrow 0$ . Take  $Q_n \in \mathcal{C}$  such that  $Q_n(A_n) = w_{\mathcal{C}}(A_n)$  for all  $n$ . Thus there exists  $\bar{\epsilon} > 0$ , and a subsequence  $n_k$  such that,  $P_{n_k}(A_{n_k}) \geq \bar{\epsilon}$  for all  $k$ . From Lemma 5 in [3],  $\mathcal{C}$  is weak-star sequentially compact thus  $\{P_{n_k}\}_k$  admits a converging subsequence  $\{P_{n_{k_l}}\}_l$ .

We may now apply a version of Vitali–Hahn–Saks [1, Theorem 8.7.4, p. 224]. Consider the probability measure  $\psi = \sum_l \frac{1}{2^l} P_{n_{k_l}}$ . Since  $\{P_{n_{k_l}}\}_l$  is weakly convergent, the set  $\{P_{n_{k_l}}\}_l$  is uniformly dominated by  $\psi$ . Moreover since  $P \gg P_{n_{k_l}}$  for all  $l$  it follows that  $P \gg \psi$ , thus  $P \gg_u \{P_{n_{k_l}}\}_l$ . Finally, as  $P(A_{n_{k_l}}) \rightarrow 0$  then  $\max_l P_{n_{k_l}}(A_{n_{k_l}}) \rightarrow 0$ , contradicting that  $P_{n_{k_l}}(A_{n_{k_l}}) \geq \bar{\epsilon}$  for all  $l$ .  $\square$

Thanks to Proposition 4.2, we may extend Remark 1 to upper integrals. Whenever  $\mathcal{C}$  is convex and weak-star compact, following Delbaen's suggestion [7, p. 224], Lemma 5 in [3] guaranties the existence of some  $P \in \mathcal{C}$  such that  $P \gg_{wk} \mathcal{C}$ .

## 5. A basic example

We finally present a natural example that illustrates the weak and strong law of large numbers for upper probabilities through limit frequencies.

**Example.** Let  $\Omega_n = \{0, 1\}$  for all  $n$  and  $\Omega = \prod_n \Omega_n$ ,  $\mathcal{A} = 2^{\Omega}$ . Define  $X_n(\omega) = \omega_n \in \{0, 1\}$  for  $\omega = (\omega_1, \dots)$ . Take  $p \in (0, 1)$ ,  $\epsilon_n \in (0, 1)$  for all  $n$  with  $\epsilon_n \leq p, 1 - p$ . Consider

$$\mathcal{C}_{\epsilon} = \left\{ P = \bigotimes_n (1 - p_n)\delta_0 + p_n\delta_1 : p_n \in [p - \epsilon_n, p + \epsilon_n] \right\} \subset \mathbf{ca}^+.$$

The standard interpretation is to consider  $\Omega$  as sequence of head and tails through independent trials with different coins and with an unknown probability lying in  $[p - \epsilon_n, p + \epsilon_n]$ .  $C_\epsilon$  is convex. We first check that  $C_\epsilon$  is weak-star closed (thus weak-star compact). Let  $\{P_\alpha\}_\alpha$  be a net in  $C_\epsilon$  converging to  $P_0$ . For  $A = \{1\} \times \prod_{m \neq n} \Omega_m$ ,  $P_\alpha(A) = p_{n,\alpha} \rightarrow p_n \in [p - \epsilon_n, p + \epsilon_n]$ .

For any cylinder  $\mathcal{I} = \prod_I \{\omega_i\} \times \prod_{-I} \Omega_n$  with  $I$  a finite subset of  $\mathbb{N}$ , we have  $P_\alpha(\mathcal{I}) = \prod_I (1 - p_{n,\alpha}) \delta_0(\omega_i) + p_{n,\alpha} \delta_1(\omega_i) \rightarrow \prod_I (1 - p_n) \delta_0(\omega_i) + p_n \delta_1(\omega_i)$ . And if  $A$  is not a cylinder then  $P_\alpha(A) = 0$  thus  $P_0(A) = 0$ . Finally,  $P_0 = \bigotimes_n (1 - p_n) \delta_0 + p_n \delta_1 \in C_h$ .

For all  $P \in C_\epsilon$  the random variables  $\{X_n\}_n$  are  $P$  independent and for all  $n$ ,

$$V_P\left(\frac{1}{n} \sum_{k=1}^n X_k\right) = \frac{1}{n^2} \sum_{k=1}^n p_n(1 - p_n) \leq \frac{1}{4n},$$

where  $p_n = P(\{1\} \times \prod_{m \neq n} \Omega_m)$ , from Theorems 4.1 and 4.2 applied to  $C_\epsilon$  we can state that for all  $\epsilon > 0$ ,

$$\lim_n w_{C_\epsilon} \left( \left\{ \frac{1}{n} \sum_{k=1}^n X_k < p - \frac{1}{n} \sum_{k=1}^n \epsilon_k - \epsilon \right\} \cup \left\{ \frac{1}{n} \sum_{k=1}^n X_k > p + \frac{1}{n} \sum_{k=1}^n \epsilon_k + \epsilon \right\} \right) = 0$$

and

$$w_{C_\epsilon} \left( \left\{ \liminf_n \frac{1}{n} \sum_{k=1}^n X_k < p - \limsup_n \frac{1}{n} \sum_{k=1}^n \epsilon_k \right\} \cup \left\{ \limsup_n \frac{1}{n} \sum_{k=1}^n X_k > p + \limsup_n \frac{1}{n} \sum_{k=1}^n \epsilon_k \right\} \right) = 0.$$

In particular if  $\lim_n \frac{1}{n} \sum_{k=1}^n \epsilon_k = 0$  then the weak law states that there exists  $N_1(\epsilon)$  such that for  $n \geq N_1(\epsilon)$ , we have

$$w_{C_\epsilon} \left( \left\{ \frac{1}{n} \sum_{k=1}^n X_k < p - \frac{1}{n} \sum_{k=1}^n \epsilon_k - \frac{\epsilon}{2} \right\} \cup \left\{ \frac{1}{n} \sum_{k=1}^n X_k > p + \frac{1}{n} \sum_{k=1}^n \epsilon_k + \frac{\epsilon}{2} \right\} \right) < \epsilon$$

now  $\frac{1}{n} \sum_{k=1}^n \epsilon_k < \frac{\epsilon}{2}$  for  $n \geq N_2(\epsilon)$ , thus for  $n \geq \max\{N_1(\epsilon), N_2(\epsilon)\}$

$$w_{C_\epsilon} \left( \left\{ \frac{1}{n} \sum_{k=1}^n X_k < p - \epsilon \right\} \cup \left\{ \frac{1}{n} \sum_{k=1}^n X_k > p + \epsilon \right\} \right) < \epsilon.$$

For the strong law it holds

$$w_{C_\epsilon} \left( \left\{ \liminf_n \frac{1}{n} \sum_{k=1}^n X_k < p \right\} \cup \left\{ \limsup_n \frac{1}{n} \sum_{k=1}^n X_k > p \right\} \right) = 0.$$

## References

- [1] K.P.S. Bhaskara Rao, M. Bhaskara Rao, *Theory of Charges*, Academic Press, New York, 1983.
- [2] O. Bondareva, Certain applications of the methods of linear programming to the theory of cooperative games, *Problem. Kibernet.* 10 (1963) 119–139 (in Russian).
- [3] A. Chateauneuf, F. Maccheroni, M. Marinacci, J.M. Tallon, Monotone continuous multiple priors, *Econom. Theory* 26 (2005) 973–982.
- [4] G. Choquet, *Theory of capacities*, *Ann. Inst. Fourier* 5 (1953) 131–295.
- [5] A. Colubi, M. Lopez-Diaz, J. Santos-Dominguez-Menchero, M. Angeles Gil, A generalized strong law of large numbers, *Probab. Theory Related Fields* 114 (1999) 401–417.
- [6] L. de Robertis, J.A. Hartigan, Bayesian inference using intervals of measures, *Ann. Statist.* 9 (1981) 235–244.
- [7] F. Delbaen, Convex games and extreme points, *J. Math. Anal. Appl.* 45 (1974) 210–233.
- [8] D. Denneberg, *Non-Additive Measure and Integral*, Kluwer, Dordrecht, 1994.
- [9] I. Gilboa, D. Schmeidler, Maxmin expected utility with non-unique prior, *J. Math. Econom.* 18 (1989) 141–153.
- [10] A.N. Kolmogorov, *Grundbegriffe der Wahrscheinlichkeitsrechnung*, Springer, Berlin, 1933, reprint, 1973; *Foundations of the Theory of Probability*, Chelsea, New York, 1956.
- [11] F. Maccheroni, M. Marinacci, A strong law of large numbers for capacities, *Ann. Probab.* 33 (2005) 1171–1178.
- [12] M. Marinacci, Limit laws for non-additive probabilities and their frequentist interpretation, *J. Econom. Theory* 84 (1999) 145–195.
- [13] M. Marinacci, L. Montrucchio, Introduction to the mathematics of ambiguity, in: I. Gilboa (Ed.), *Uncertainty in Economic Theory: A Collection of Essays in Honor of David Schmeidler's 65th Birthday*, Routledge, New York, 2004, pp. 46–107.
- [14] M.L. Puri, D.A. Ralescu, Strong law of large numbers for Banach space valued random sets, *Ann. Probab.* 11 (1983) 222–224.
- [15] M.L. Puri, D.A. Ralescu, Strong law of large numbers with respect to a set-valued probability measure, *Ann. Probab.* 11 (1983) 1051–1054.
- [16] D. Ralescu, G. Adams, The fuzzy integral, *J. Math. Anal. Appl.* 75 (1980) 562–570.
- [17] L.S. Shapley, On balanced sets and cores, *Naval Res. Logist. Quart.* 14 (1967) 453–460.
- [18] D. Schmeidler, Cores of exact games, *J. Math. Anal. Appl.* 40 (1972) 214–225.
- [19] D. Schmeidler, Integral representation without additivity, *Proc. Amer. Math. Soc.* 97 (1986) 255–261.
- [20] M. Sugeno, *Theory of fuzzy integrals and its applications*, PhD thesis, Tokyo Institute of Technology, Tokyo, 1974 (in Japanese).